Analytical tableaux for da Costa's paraconsistent logics[1]

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In 1963 da Costa (see da Costa (1974)) introduces his hierarchies of logical calculi for the study of inconsistent but non-trivial theories: the hierarchy of propositional calculi $\boldsymbol{C}_n$, $1 \leq n \leq \omega$, the hierarchy of predicate calculi $\boldsymbol{C}_n^*$, $1 \leq n \leq \omega$, the hierarchy of predicate calculi with equality $\boldsymbol{C}_n^=$, $1 \leq n \leq \omega$, and the hierarchy of calculi of descriptions $\boldsymbol{D}_n$, $1 \leq n \leq \omega$.

Marconi (1980) introduces a variant of semantical tableaux systems, *à la* Beth (see Beth (1959)), in order to prove the completeness and decidability of da Costa's propositional system $\boldsymbol{C}_1$. He also claims that his method can be expanded for the systems $\boldsymbol{C}_n$, $2 \leq n < \omega$. The system introduced by Marconi is based on the same intuitions underlying the definition of quasi-matrices introduced by da Costa and Alves (1976), simplifying the verification process of validity of the formulae. In Marconi's tableaux system the rules for the connectives &, $\vee$ and $\supset$ are the standard ones, and two special rules are added to operate with the paraconsistent negation.

Alves (1976) introduces the propositional paraconsistent system $\boldsymbol{C}_1^1$, by replacing the schema of axioms $\neg\neg A \supset A$ of $\boldsymbol{C}_1$ by the schema $\neg\neg A \equiv A$, in order to obtain a system stronger than da Costa's $\boldsymbol{C}_1$. Carnielli and Lima-Marques (1992) introduce a semantical tableaux system, *à la* Smullyan (see Smullyan (1968)), for Alves's paraconsistent propositional logic $\boldsymbol{C}_1^1$ and for the paraconsistent quantificational logic with equality $\boldsymbol{C}_1^{1=}$, namely the systems $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^=$ respectively, and show that these systems are complete and decidable.

Buchsbaum and Pequeno (1993) introduce a syntactical tableaux system, also *à la* Smullyan, for da Costa's $\boldsymbol{C}_1^*$, the system *S***C**1*, showing that *S***C**1* is complete.

Castro (1998) and Castro and D'Ottaviano (2000) study a hierarchy of natural deduction systems, namely $\boldsymbol{NDC}_n$, $1 \leq n \leq \omega$, through Fitch´s method of subordinate proofs (see Fitch (1952)). In spite of the main syntactical and semantical results being natural consequences of the logical equivalence between the hierarchies $\boldsymbol{C}_n$ and $\boldsymbol{NDC}_n$, $1 \leq n \leq \omega$, we directly prove the soundness and completeness of the systems from Loparić and Alves' semantics (see da Costa and Alves (1976, 1977) and Loparić and Alves (1980)).

Castro (2000, 2004) and D'Ottaviano and Castro (2005), based on the systems $\boldsymbol{NDC}_n$, $1 \leq n < \omega$, and by using the method of analytical tableaux (see Smullyan (1968) and van Fraassen (1971)), introduce a hierarchy of syntactical tableaux systems $\boldsymbol{TNDC}_n$, $1 \leq n < \omega$, in which every system $\boldsymbol{TNDC}_n$ is equivalent to da Costa's corresponding system $\boldsymbol{C}_n$, $1 \leq n < \omega$. In particular, our $\boldsymbol{TNDC}_1$ is distinct of Marconi's formulation, of Carnielli and Lima-Marques's tableaux system $\boldsymbol{TC}_1$ and of Buchsbaum and Pequeno's tableaux formulation *S***C**1. We prove a generalized Cut Rule (Theorem) for the systems $\boldsymbol{TNDC}_n$, $1 \leq n < \omega$. Then, we prove that each system of this hierarchy is logically equivalent to the corresponding paraconsistent system $\boldsymbol{C}_n$, $1 \leq n < \omega$.

In Castro and D´Ottaviano's tableaux systems $\boldsymbol{TNDC}_n$, $1 \leq n < \omega$, we introduced da Costa's defined 'ball' operator 'o', *plus* the generalized operators 'k' and '(k)', and the negations '$\sim_k$', for $k \geq 1$, as primitive operators. As far as we know, there is no other tableaux system in the literature, for any one of da Costa's systems $\boldsymbol{C}_n$, $1 \leq n < \omega$, in which the mentioned da Costa's operators are introduced as primitive.

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As far as we know in the literature, the first paraconsistent system in which da Costa's 'ball' operator 'o' was treated as a primitive "consistency" operator was introduced by D´Ottaviano and Epstein (1988), in a modified version of the system $\mathbf{J}_3$ of D´Ottaviano and da Costa (1970), also presented in Epstein (1990, 1995), Chapter IX, written in collaboration with D'Ottaviano. In some recent papers (see Carnielli, Coniglio and Marcos (2007)) several paraconsistent axiomatic (Hilbert style) propositional systems have been studied by considering da Costa's 'ball' operator 'o' as a primitive operator: the system $\mathbf{J}_3$ and da Costa's propositional systems $\boldsymbol{C}_n$, $1 \leq n < \omega$, are introduced by the authors as particular cases of these paraconsistent systems.

Our systems $\boldsymbol{TNDC}_n$, as the other mentioned tableaux systems for da Costa's calculi, constitute automated theorem proving systems.

In the system *SC*1* of Buchsbaum and Pequeno we do not have an explicit rule that determines *a priori* when the definition of the operator 'o' must be used or must not be used during the derivations; on account of this it is possible to occur open branches that must be rebuilt, in a distinct way, from the mentioned occurrence of the operator 'o'.

Also in Carnielli and Lima-Marques's systems $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^{=}$ there are not specific rules that determine *a priori* when to use the definition of the defined operator 'o', what may make necessary to rebuild branches. Particularly, in these systems infinite loops may occur, 'postponing indefinitely', according to the own authors, the analysis of formulae that involve the operator of primitive negation and, as a natural consequence, the operator 'o'; Carnielli and Lima-Marques prove the decidability of $\boldsymbol{TC}_1$ and $\boldsymbol{TC}_1^{=}$, showing how to deal with the infinite loops. Carnielli, Coniglio and Marcos (2007) improve the system $\boldsymbol{TC}_1$, introducing a new semantical tableaux system for $\boldsymbol{C}_1$, trying to avoid the presence of infinite loops in the derivations of the branches: in this new tableaux system da Costa's 'ball' operator 'o' is maintained as a defined operator and, as the nodes of the branches in the derivations are not univocally determined, it is possible (as in Buchsbaum and Pequeno's *SC*1* system) to occur open branches that may be rebuilt in a distinct way.

Due to the " primitiveness" of the 'ball' operator 'o' and of the other denumerable above mentioned da Costa's operators, in every one of our $\boldsymbol{TNDC}_n$, $1<n<\omega$, there are specific rules to objectively deal with the operator 'o', as well as with the operators 'k' and '(k)', for $k \geq 1$. The branches of the tableaux are univocally and automatically generated and infinite loops do not occur. Another peculiarity of our tableaux systems is that, differently to what is in the literature, we define two conditions for the closure of the branches of the tableaux of $\boldsymbol{TNDC}_n$, for every n, $1 \leq n < \omega$: either they are closed by the strong negation '$\sim_n$', as usual, or they are closed by the paraconsistent negation '$\neg$' and additional conditions.

In this paper, motivated by our previous works, we introduce a new hierarchy of quantificational analytical tableaux systems, the $\boldsymbol{TNDC}_n$*, $1 \leq n < \omega$. The operator 'o' is primitive, as well as the operators for negation '$\sim_k$' and the connectives 'k' and '(k)', for any $k \geq 1$; as in the case of our previous paper, it is necessary to deal with specific problems concerning relationships between the generalized distinct primitive operators; and with relationships between the different systems of the hierarchy $\boldsymbol{TNDC}_n$*, $1 \leq n < \omega$. We also define two conditions for the closure of the branches of the tableaux $\boldsymbol{TNDC}_n$*, $1 \leq n < \omega$.

We prove a version of Cut Rule (Theorem) for the systems $\boldsymbol{TNDC}_n$*, $1 \leq n < \omega$, and also prove that these systems are logically equivalent to the corresponding systems $\boldsymbol{C}_n$*, $1 \leq n < \omega$, respectively.

We observe that every one of our systems $\boldsymbol{TNDC}_n$*, $1 \leq n < \omega$, is introduced from a denumerable (infinite) set of primitive operators, what finally allows us to capture da Costa's systems $\boldsymbol{C}_n$*, $1 \leq n < \omega$, as paraconsistent extensions of first-order classical predicate logic.

The systems $\boldsymbol{TNDC}_n$* constitute completely automated theorem proving systems for da Costa's logical systems $\boldsymbol{C}_n$*, $1 \leq n < \omega$.

As far as we know, besides considering da Costa's families of operators as primitive operators, this is the first paper in the literature in which all the systems of the da Costa´s hierarchy of quantificational paraconsistent systems $\boldsymbol{C}_n$*, $1 \leq n < \omega$, receive a tableaux approach.

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